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Organization measures and representations of the Kohonen maps

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Abstract. The Kohonen's algorithm is known and used to map automatically an input space with a grid of neurons. When this space is high-dimensional, it becomes difficult to analyse the state of the network, because it is no longer possible to represent the network in the weight space. In this paper, we enumerate some existing techniques used to analyse the network state and we show their limitations. We also present a new method that provides useful information about the organization degree even in high-dimensional spaces.

Keywords: Kohonen network, high-dimensional spaces, organization.

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1 Introduction

In 1982, T. Kohonen has proposed an original neuronal algorithm which realizes the quantization of an input space that can be continuous or discrete and of arbitrary dimension [4]. Inspired by biological observations on the self-organization (especially on the retinotopic maps formation mechanisms), this algorithm does not intend to modelize all the phenomenons that provide this self-organization, but rather try to emphasize the main caracteristics of this function: it realizes the non-supervised learning of the projection from a stimuli space toward a "cortical" map of neurons, conserving the neighbourhood relations and depending on the statistics of the stimuli. Since this learning should be seen as the placement of the neuron weights vectors in the input space in respect of its topology, we also call this algorithm "features extraction".

The potential and existing applications of this algorithm are numberous and concern several areas, as robotic, image processing, process control and in general data analysis. However, despite of the wide usage of this algorithm, there is only few methods to analyse the network configuration while and after learning. Since the algorithm is very sensible to the different learning parameters, to the initial configuration, to the nature of the stimuli and to their mode of presentation, this lack of analyze methods is very annoying.

In the following sections, we show some methods to analyse the network convergence and we propose a new one that emphasizes the degree of organization.

2 The model

We consider here the simplified algorithm [4]:

Let an input vector $\mathbf{x}$ have n components $[x_1, \ldots, x_n]^T$, as well as the weight vectors of all the neurons i ($1 \leq i \leq N$), i.e. $\boldsymbol{W}_i\ [W_{i1}, \ldots, W_{in}]^T$. Using a distance measure $\delta(\mathbf{x}, \boldsymbol{W}_i)$ between $\mathbf{x}$ and $\boldsymbol{W}_i$ (for example the euclidean distance), the index k of the neuron with the best response to an input vector $\mathbf{x}$ is given by the condition:

$$\delta(\mathbf{x}, \boldsymbol{W}_k) = \min \delta(\mathbf{x}, \boldsymbol{W}_k)\ , \quad 1 \leq i \leq N \tag{1}$$

Then, around this winning neuron k, we define a subset $V_k(t)$ of neurons within a neighbourhood. The weights of this neuron subset are adapted as following:

$$\boldsymbol{W}_i(t+1) = \boldsymbol{W}_i(t) + \alpha(t)\left[\mathbf{x}(t) - \boldsymbol{W}_i(t)\right] \ , \quad i \in V_k(t) \tag{2}$$

The weights of the other neurons remain unchanged.

Let us remark that the neibourhood radius is a decreasing function of the discrete time t, as well as the gain parameter α.

The repetition of these steps (we do them for each presented stimulus $\mathbf{x}$) conducts the weights $\boldsymbol{W}_i$ to converge to a discretized image of the stimuli density distribution. This convergence is demonstrated in the case of a monodimensional network, with a uniform [2] or non-uniform [1] input distribution.

To formalise the meaning of the term "organized" when the input and weight space dimension is arbitrarily high, we say that a Kohonen map is well *organized* when both of the following conditions are satisfied:

1. There is a correct *quantization* of the input space (every neuron has about the same probability to be excited). This implies a concentration of neurons in the regions of the input space where the stimuli are frequent.

2. There is a topological conservation of the input space (neighbourhood relations are conserved by the projection in the Kohonen map). This implies *ordered* neurons.

3 Usual representations and measures

3.1 Network representation

The most known representation of the Kohonen network is the weight positionning of the neurons on a plane (we use the weights vectors as the coordinates of the grid intersections). This representation shows immediately and intuitively the organization quality as previously defined (section 2).

Unfortunately, when the input space dimension is greater than 2, it is necessary to make a projection on two dimensions to represent the grid. The lack of information is generally unacceptable when selecting only 2 components in the weight vectors.

In the section 4, we describe a representation that overcomes this major difficulty.

3.2 Measures

In [6], a new (and faster) variant of the algorithm is presented. The simulations are analysed through the evaluation of the variance of firing rate of the neurons. When the neurons quantize well the input space (condition (1) of the section 2), this variance tends to 0, because each neuron has the same probability to be excited. On the contrary, a great variance in the neurons excitation is the indication of a bad quantization. So, this measure gives a suitable information about the quantization, but does not take in consideration the good topology conservation (the order of the units). A simple example can illustrate this fact: If the neighbourhood is fixed to 0 from the beginning, the algorithm degenerates to a simple quantization algorithm, without any topological conservation (because without any order in the units). Indeed, the place of weights vectors in the input space is not influenced by the place of the neurons in the grid, because the neighbourhood in the grid has never been used during the learning. However, the excitation variance measure will not signal this lack of organization in the sense of our definition.

Another study [7] concerns the analysis of the equilibrium state reached by the network after convergence. An equation describing the stationnary state of the network is given, from where the expression of the local magnification factor is derived. This magnification factor is computed in the case of a monodimensional input and weights space, but the authors indicate that such a development cannot be driven within two dimensions.

Another measure, which does not take into account the input distribution, consists in the number of inversions in a mono-dimensional network (the grid and the input and weight space are in one dimension). Proposed in [5], this measure is used in [2] to demonstrate the convergence of the network in a state where the units are ordered. By its definition itself, this measure is unapplicable for many dimensions.

4 "Curvilinear" representation

In [3], we introduce a "curvilinear" representation which looks like the weight positioning representation, but may be used with an arbitrary number of dimensions. The idea consists in the "unfolding" of the surface described by the grid of neurons in the weights space. The neurons are positionned on a plane using their curvilinear coordinates in the surface, i.e. in function of their respective distances in the weights space. In practice, we position first the neurons of the central axes of the grid (w: width of the grid, h: height, neuron ij position: $\boldsymbol{P}^{ij}$) :

$$P_x^{h/2,j} = P_y^{i,w/2} = 0 \tag{3}$$

Then, the position of the other neurons is computed recursively from these axes to the grid borders (for example here for the upper right quarter of the grid, the expression for the other quarters should be found by symetry):

$$\begin{aligned} P_x^{ij} &= P_x^{i,j-1} + \|\boldsymbol{W}_{ij} - \boldsymbol{W}_{i,j-1}\| && (4) \\ P_y^{ij} &= P_y^{i-1,j} + \|\boldsymbol{W}_{ij} - \boldsymbol{W}_{i-1,j}\| && (5) \end{aligned}$$

Figure 1 illustrates this representation. We show the learning phase of a 30×10 network in a 3D space with a "horseshoe" shape input distribution. The upper images show the usual weight positioning representation, while the lower ones show the equivalent in the curvilinear representation.

5 Organization measure

5.1 The "Θ" measure

In [3], we define also a scalar organization measure that represents the disorder degree of the grid. This measure Θ is at a given time the ratio between the standard deviation of the distances between consecutive neurons weights vectors and the mean of these distances:

$$\Theta = \frac{\sigma_\Delta}{\mu_\Delta} \tag{6}$$

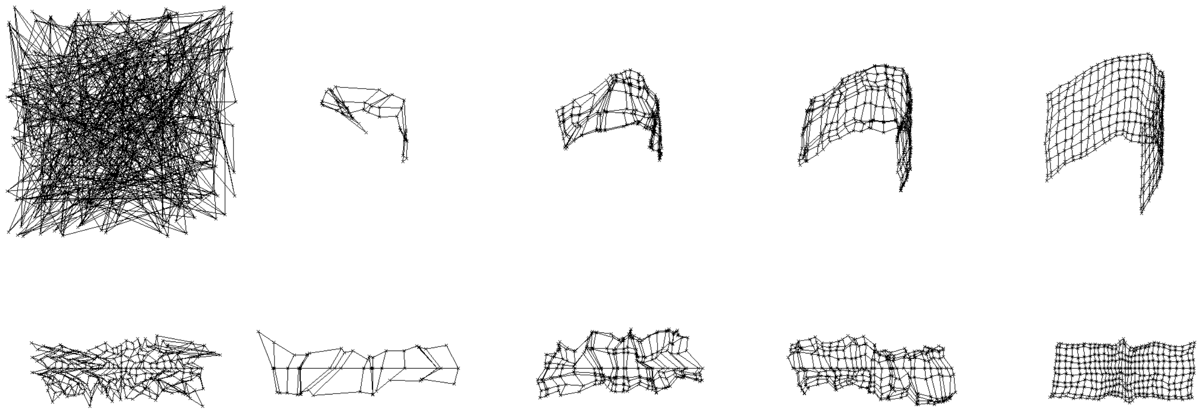

Figure 1: The usual weight positioning representation (up) and the curviliear representation (down) for a learning sequence with a "horseshoe" shape input distribution in a 3D space.

With μ_Δ the mean of the euclidean distances between the weight vectors of two neighbour neurons, and σ_Δ the standard deviation of these distances. If the grid is absolutely regular, Θ tends to zero. On the contrary, more the grid is disordered, more Θ gets great.

This measure is scalar and so it should be plotted in function of time. We observe in a first time a growth of the curve corresponding to the network ordering. In a second time, the Θ function decreases while the network grid regularizes [3].

5.2 "δ_i, δ_w" relation

We want to generalize this idea considering not only the distances between two consecutive neurons, but between all the neurons of the Kohonen map. We focus on the relation between the physical (or grid index) distance δ_i between two neurons and their weights distance δ_w. Since the network is discrete, the physical distance δ_i in the grid is stepwise, and it can only take the values $\left\{1; \sqrt{2}; 2; \sqrt{5}; \ldots\right\}$, while the weights distance is a continuous value. In the right part of the figures 2 and following, we show the distribution of the couples (δ_i, δ_w) for several cases; in the central part of the figures, we show the mean of δ_w for each δ_i, as well as the mean plus (or minus) the standard deviation; the left part contains the usual weight positioning representation. In the figure 2, we show this representation in two cases. Up: just after the weights initalisation to random values. Down: after 10000 iterations with a 2D uniform distribution in $[0,1]^2$. With the random weights, the mean of δ_w for each δ_i is horizontal (central part of the figure). It means that no particular organization is present. On the contrary, when the grid is regular, this mean follow a growthing line, indicating that the distance between the weights of two units is proportional to the physical distance of these units in the Kohonen grid. This gives information about the condition (2) ment by "good organization" (section 2).

The interest of this representation is that it provides an information about the units order, and this whatever high the input and weight space dimension is.

We illustrate this representation in the figures 4a to 4n, where simple distributions are mapped with more or less success. These figures are commented in the section 7.

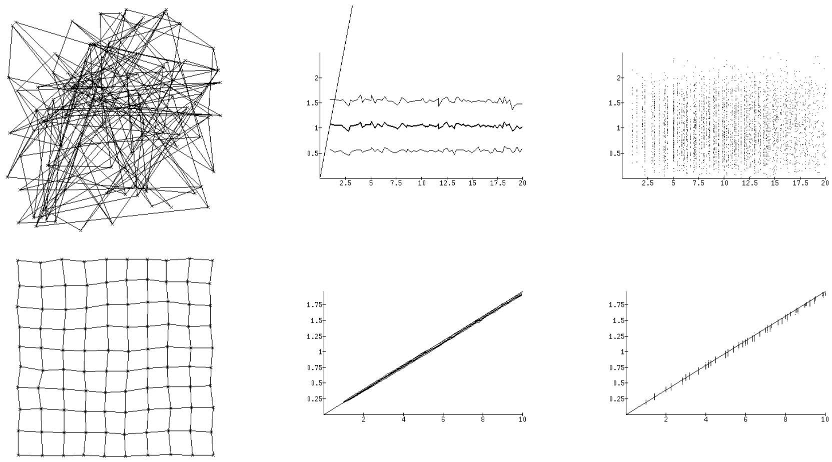

Figure 2: The "δ_i, δ_w" relation shown in two cases: Up: with random weights. Down: after 10000 iterations with a square distribution: the network is ordered.

6 15 dimensions example

In view to validate this representation, we consider the organization of a bi-dimensional network whose input and weights space is in n dimensions ($n > 2$). We compare the network self-organization in function of two distinct stimuli distributions. To generate the first distribution, we simulate a system whose the purpose is to measure the position of an ultrasonic source on a plane. Several (n) sensors are randomly disposed on the plane and give the distance to the source. We collect all the n distances in a vector, constituting the first input distribution. Such a system could learn in a non-supervised way the n to 2 coordinates transform, with no need to know the position of the n sensors. Although the vectors dimension is arbitrarly high (n), the *freedom degree* of the system is naturally only 2 (because the source is on a plane).

On the contrary, the second distribution is uniform in $[0,1]^n$. Since the n components are independant, the freedom degree of this distribution is n.

The results obtained with the first or the second input distribution with a network of 20×20 neurons and $n = 15$ dimensions are shown below in figure 3. In the left part of the figure, the curvilinear representation (cf. section 4) is approximatively similar for both cases, which gives false impression that the organization is comparable. Also, the excitation variance measure as described in the section 3 (but not shown here), does not signal any organization quality difference between both cases. However, it is obvious that, in the case of the uniform distribution, the neurons grid will describe many folds in the weight space, and that the topology conservation will be only *local*. On the contrary, the topology conservation is larger when the distribution stems from the sensors system.

The "δ_i, δ_w" relation emphasizes this fact, as it is shown in the middle and the right parts of figure 3. *Organization length*, that we define as the mean number of units disposed approximatively linearly, does not excess 5 in the case of uniform distribution (for $\delta_i > 5$, the mean of δ_w is horizontal). On the other hand, the sensors distribution give a network were the topological conservation remains on the whole network.

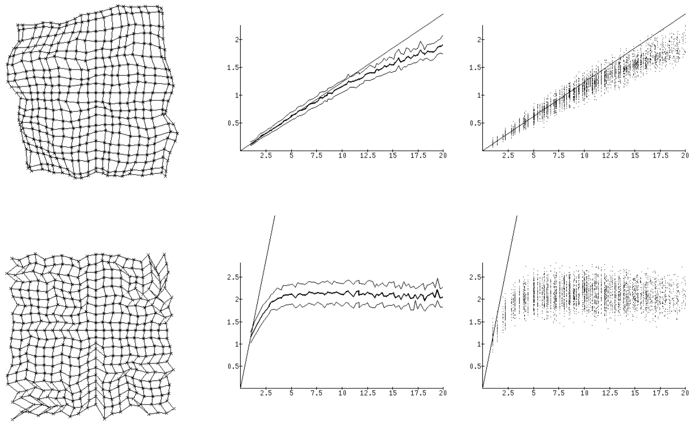

Figure 3: Representation of the "δ_i, δ_w" relation (right and middle part) and the curvilinear representation (left part) for a 20×20 network after self-organisation. The input distribution has 15 dimensions. Up: the distribution stems from the sensors system (see text). Down: the distribution is uniform for each dimension.

7 Discussion

In figures 4a to 4n, we illustrate the "δ_i, δ_w" relation for several well identified 2D or 3D distribution cases.

In figures 4b, 4c and 4d, we should observe that the general shape of the distribution does not affect too much the mean of $\delta_w(\delta_i)$, which remains approximatively a straight line, even if the standard deviation is greater than those obtained in the case of the prefectly regular grid.

In figure 4e, the input distribution was non-uniform (coordinates $rnd^2(0,1)$ and random sign). The mean of $\delta_w(\delta_i)$ remains growing, but is less straight than in the case of the perfect grid.

Figures 4g, 4h et 4i are particularly interesting, because they show what happens when the dimension of the neurons grid is less than the number of freedom degrees. In this case, the distribution is in 2 dimensions, while the network grid is only monodimensional. The "δ_i, δ_w" relation emphasizes the folds of the network. Also, we should remark that, despite of the neurons number (and so folds number) raise, the general shape of the "δ_i, δ_w" points cloud remains approximatively the same.

The same remark should be done for figures 4j and 4k, where the distribution was a kind of propeller. This time, the mean remains growing more longer.

In figures 4l et 4m, the distribution was bimodal. Then, the mean and the standard deviation curves do not give suitable information anymore. However, the "δ_i, δ_w" points clouds emphasize the bimodality of the distribution. Let us remark also that the lower part of the clouds looks like the complete cloud obtained with the same distribution, but within only one mode (the lower part of the cloud of fig. 4l looks like those of figs. 4g, 4h et 4i, and the lower part of the cloud of fig. 4m looks like those of fig. 4b).

Finally, figure 4n shows the "δ_i, δ_w" relation obtained with the "horseshoe" 3D distribution. The "organization length" seems here not to excess 8, because of the fold of the distribution.

8 Conclusion

The analysis of the Kohonen network which self-organizes in a high-dimensional space is difficult. Indeed, there is no direct method to observe the network as when the input space is only in 2 dimensions. The diversity of the potential applications of this network is very large, but we are sure that a lot of possible users hesitate considering the relative lack of methods to analyse the network organization quality. That is the reason why we feel the necessity to develop such methods, as those ones discribed in this paper.

We have described several methods already used by other works, in particular the measure of the neurons excitation variance, which shows how the set of neurons quantizes the input space. This measure does not give any information about the other fundamental property of the Kohonen self-organization algorithm, i.e. the unit order that allows a neighbourhood relations conservation between the stimuli and the neurons.

So we proposed a complementary method which emphasizes the quality of this order (good quality or bad one). On the other hand, this method does not give any information about the quantization quality, because it does not take into account the input distribution.

Besides, we believe that only a set of several measures and representations complementar to each others will provide an exhaustive information on the network state and its convergence.

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Figure 4: Examples of the representation of the weight distances in function of the Kohonen grid unit index distance for several network configurations (see text).

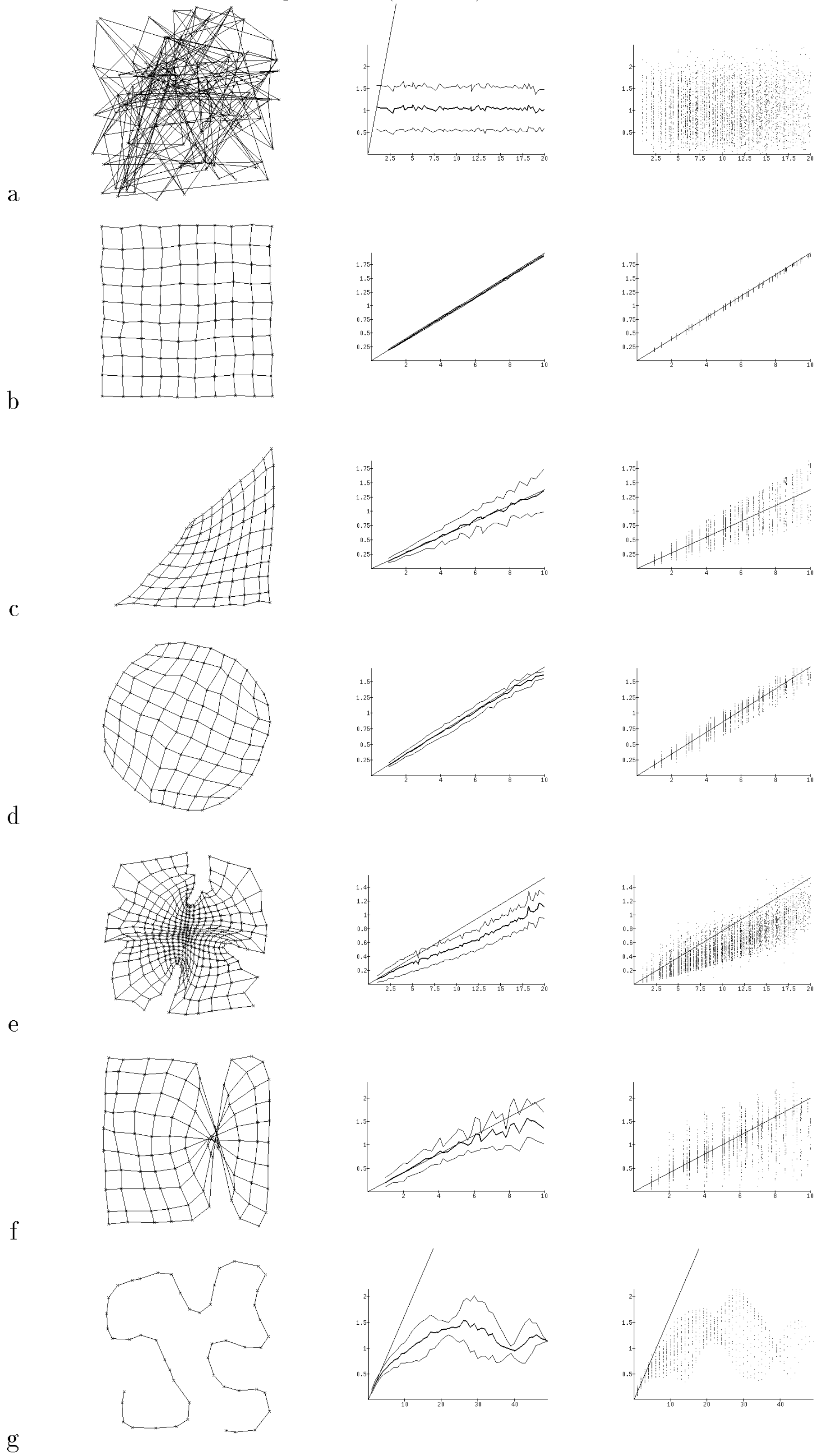
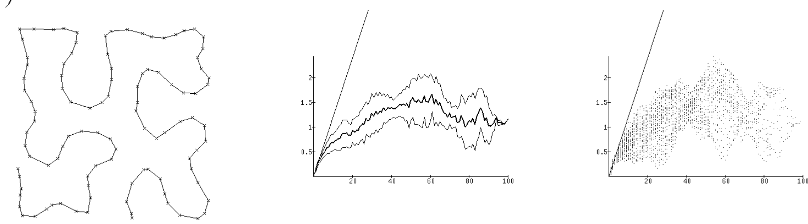


Figure 4 (continued)



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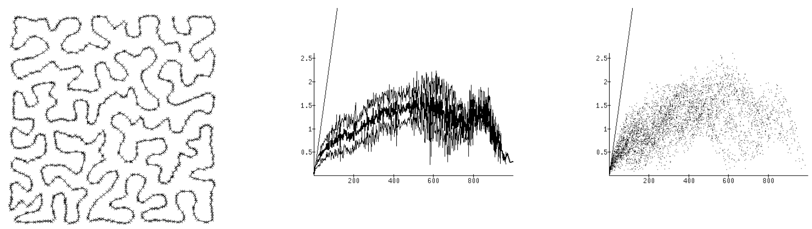

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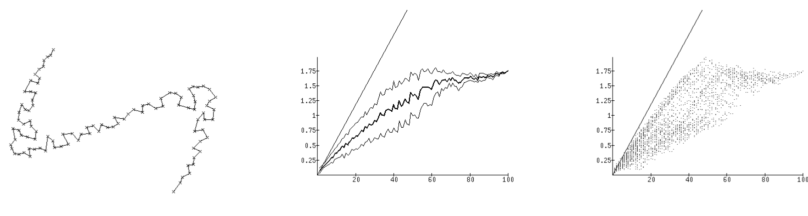

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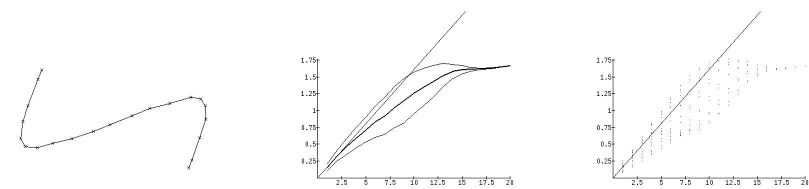

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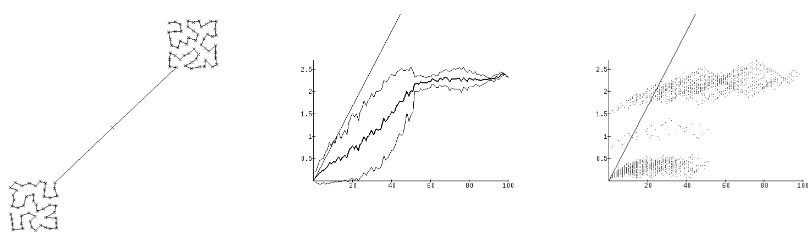

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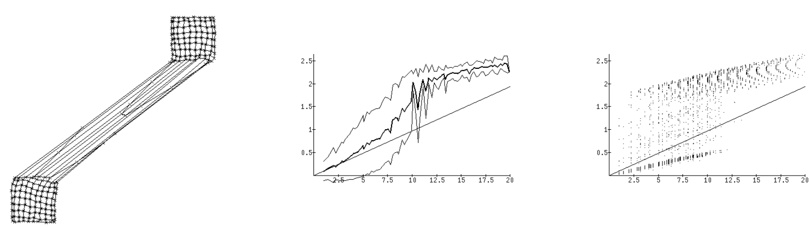

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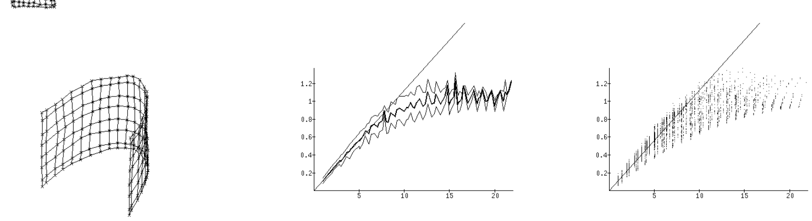

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